

Lie group analysis for short pulse equation

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January 18, 2012

Abstract

In this paper, the classical Lie symmetry analysis and the generalized form of Lie symmetry method are performed for a general short pulse equation. The point, contact and local symmetries for this equation are given. In this paper, we generalize the results of H. Liu and J. Li [1], and add some further facts, such as optimal system of Lie symmetry subalgebras and two local symmetries.

Keywords: Short pulse equation, Lie symmetry analysis, Point, contact and local symmetries.

1 Introduction

Nonlinear PDEs arising in many applied fields like the biology, fluid mechanics, plasma physics and optics, systems of impulse and neural networks, etc, and exhibit a rich variety of nonlinear phenomena. The investigation of the exact solutions plays an important role in the study of nonlinear systems. In this paper, we find Lie point symmetries, third order local symmetries, optimal system of these two type symmetries, and corresponding invariant solutions for a general short pulse equation:

$$\text{SPE} : u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx} \quad (1)$$

where $u = u(x, t)$ is the unknown real function and subscripts denote differentiation w.r.t. x and t ; α and β are nonzero real parameters.

This general SPE was derived by T. Schafer and C.E. Wayne [2, p.94] as a model equation describing the propagation of ultra-short light pulses in silica optical fibres. In [2, 3], many results are obtained about the special SPE:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \quad (2)$$

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2 Lie contact and point symmetries

Let $J^1 = J^1(\mathbb{R}^2, \mathbb{R})$ be the jet space with coordinates (x, t, u, u_x, u_t) . Let

$$\mathbf{v} = \xi \partial_x + \tau \partial_t + \eta \partial_u + \eta^x \partial_{u_x} + \eta^t \partial_{u_t} \quad (3)$$

be an infinitesimal Lie contact symmetry of (1), where ξ , τ and η are functions $J^1(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}$, and

$$Q = \xi u_x + \tau u_t - \eta \quad (4)$$

be the characteristic function of (4). Thus

$$\begin{aligned} \xi &= Q_{u_x}, \quad \tau = Q_{u_t}, \quad \eta = u_x Q_{u_x} + u_t Q_{u_t} - Q, \\ \eta^x &= -Q_x - u_x Q_u, \quad \eta^t = -Q_t - u_t Q_u. \end{aligned} \quad (5)$$

Then, the (3) is an infinitesimal Lie contact symmetry of the (1) if and only if

$$\mathbf{v}^{(2)} \left(u_{xt} - \alpha u - \frac{1}{3} \beta (u^3)_{xx} \right) = 0, \quad u_{xt} = \alpha u + \frac{1}{3} \beta (u^3)_{xx}, \quad (6)$$

where $\mathbf{v}^{(2)}$ is the second prolongation of \mathbf{v} :

$$\mathbf{v}^{(2)} = Q \partial_u + D_x Q \partial_{u_x} + D_t Q \partial_{u_t} + D_x^2 Q \partial_{u_{xx}} + D_x D_t Q \partial_{u_{xt}} + D_t^2 Q \partial_{u_{tt}}, \quad (7)$$

where D_x and D_t are total derivative w.r.t x and t , respectively.

By substituting $u_{x,t}$ from second equation of (7) in the first equation, we find a polynomial of u_{xx} and u_{tt} with functional coefficients of (x, t, u, u_x, u_t) . Its coefficients must be zero:

$$\begin{aligned} Q_{u_x, u_x} &= 0, \quad Q_{u_x, u_t} = 0, \quad Q_{u_t, u_t} = 0, \\ u_t Q_{u, u_x} + \alpha u Q_{u_x, u_x} + Q_{t, u_x} &= 0, \quad \alpha u Q_{u_t, u_t} + u_x Q_{u, u_t} + Q_{x, u_t} = 0, \\ u_x^2 Q_{u, u} + 2u_x Q_{x, u} + Q_{xx} + \alpha u (\alpha Q_{u_t, u_t} + 2u_x Q_{u, u_t} + 2Q_{x, u_t}) &= 0, \\ u_t Q_{u, u_t} - 5u_x Q_{u, u_x} - 5Q_{x, u_x} + Q_{t, u_t} - 4u Q_{u_x, u_t} - 2u_x Q_u &= 0, \\ u_x u_t Q_{u, u} + u_x Q_{t, u} + u_t Q_{x, u} + Q_{x, t} \\ + \alpha u (u_t Q_{u, u_t} + Q_{t, u_t} + Q_{x, u_x} + Q_u) + \alpha (u_x Q_{u_x} + u_t Q_{u_t} - Q) &= 0. \end{aligned} \quad (8)$$

After solving the determining system (8), one finds that

$$Q = C_1 u_x + C_2 u_t + C_3 (xu_x - tu_t - 3u); \quad (9)$$

where, C_1 , C_2 and C_3 are arbitrary constants. Therefore,

Theorem *The SPE (1) has a 3-dimensional Lie algebra \mathfrak{g} of point symmetries, generated by the infinitesimal generators*

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = x \partial_x - t \partial_t + 3u \partial_u, \quad (10)$$

and commuting table

$[\cdot, \cdot]$	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3
\mathbf{v}_1	0	0	\mathbf{v}_1
\mathbf{v}_2	0	0	$-\mathbf{v}_2$
\mathbf{v}_3	$-\mathbf{v}_1$	\mathbf{v}_2	0

(11)

The SPE (1) has not any non-point contact symmetry.

3 Invariant solutions and its classification

The one-parameter groups G_i generated by the base of \mathfrak{g} are as follows:

$$\begin{aligned} G_1 &: \exp(\varepsilon \mathbf{v}_1) \cdot (x, t, u) = (x + \varepsilon, t, u), \\ G_2 &: \exp(\varepsilon \mathbf{v}_2) \cdot (x, t, u) = (x, t + \varepsilon, u), \\ G_3 &: \exp(\varepsilon \mathbf{v}_3) \cdot (x, t, u) = (e^\varepsilon x, e^{-\varepsilon} t, e^{3\varepsilon} u), \end{aligned} \quad (12)$$

where ε is a real number.

Since each group G_i is a symmetry group of SPE (1) and if $u = f(x, y)$ is a solution of the SPE (1), so are the following functions

$$u = f(x + \varepsilon, t), \quad u = f(x, t + \varepsilon), \quad u = f(e^\varepsilon x, e^{-\varepsilon} t, e^{3\varepsilon} u), \quad (13)$$

where ε is an arbitrary real number. Thus, for the arbitrary combination $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \in \mathfrak{g}$, the SPE (1) has the following solution:

$$u = f(e^{\varepsilon_3} x + \varepsilon_1, e^{-\varepsilon_3} t + \varepsilon_2, e^{-3\varepsilon_3} u), \quad (14)$$

where ε_i are arbitrary real numbers.

Let G be the symmetry Lie group of SPE (1). Now G operates on the set of solutions S of SPE (1), and $s \cdot G$ be the orbit of s , and H be a subgroup of G . Invariant H -solutions $s \in S$ are characterized by equality $s \cdot S = \{s\}$. If $h \in G$ is a transformation and $s \in S$, then

$$h \cdot (s \cdot H) = (h \cdot s) \cdot (h H h^{-1}). \quad (15)$$

Consequently, every invariant H -solution s transforms into an invariant $h H h^{-1}$ -solution (Proposition 3.6 of [4]). Therefore, different invariant solutions are found from similar subgroups of G . Thus, classification of invariant H -solutions is reduced to the problem of classification of subgroups of G , up to

similarity. An optimal system of s -dimensional subgroups of G is a list of conjugacy inequivalent s -dimensional subgroups of G with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s -dimensional subalgebras forms an optimal system if every s -dimensional subalgebra of \mathfrak{g} is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \text{Ad}(g) \cdot \mathfrak{h}$. Let H and \tilde{H} be connected, s -dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras \mathfrak{h} and $\tilde{\mathfrak{h}}$ of the Lie algebra \mathfrak{g} . Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathfrak{h}} = \text{Ad}(g) \cdot \mathfrak{h}$ are conjugate subalgebras (Proposition 3.7 of [4]). Thus, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so we concentrate on it.

For the one-dimensional subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in Lie algebra symmetries of SPE (1) and so to "simplify" it as much as possible. The adjoint action is given by the Lie series

$$\text{Ad}(\exp(\varepsilon \mathbf{v}_i) \mathbf{v}_j) = \mathbf{v}_j - \varepsilon [\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2} [\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \quad (16)$$

where $i, j = 1, \dots, 3$. Let $F_i^\varepsilon : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by $\mathbf{v} \mapsto \text{Ad}(\exp(\varepsilon \mathbf{v}_i) \mathbf{v})$, for $i = 1, \dots, 3$. Therefore, if $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \in \mathfrak{g}$, then

$$\begin{aligned} F_i^{\varepsilon_1}(\mathbf{v}) &= (c_1 + \varepsilon_1 c_3) \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3, \\ F_i^{\varepsilon_2}(\mathbf{v}) &= c_1 \mathbf{v}_1 + (c_2 + \varepsilon_2 c_3) \mathbf{v}_2 + c_3 \mathbf{v}_3, \\ F_i^{\varepsilon_3}(\mathbf{v}) &= e^{-\varepsilon_3} c_1 \mathbf{v}_1 + e^{\varepsilon_3} c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3. \end{aligned} \quad (17)$$

Applying these transformations, one can show that

Theorem *An one-dimensional optimal system of \mathfrak{g} is*

$$\mathbf{v}_1 + a \mathbf{v}_2, \quad b \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_3, \quad (18)$$

where a and b are real constants; and, a two-dimensional optimal system of \mathfrak{g} is given by

$$\mathbf{v}_1, \mathbf{v}_2, \quad \mathbf{v}_1, \mathbf{v}_3, \quad \mathbf{v}_2, \mathbf{v}_3. \quad (19)$$

4 Local symmetries of SPE

One can generalize one-parameter Lie groups of point transformations with infinitesimal generators in the characteristic form $\mathbf{v} = Q(x, t, u, u_x, u_t) \partial_u$ to

one-parameter s -order local transformations with infinitesimal generators of the form

$$\mathbf{v} = Q(x, t, u, \partial u, \partial^2 u, \dots, \partial^s u) \partial u, \quad (20)$$

where the infinitesimal components depend on derivatives of u up to some finite order $s \geq 1$. The prolongation of \mathbf{v} is given by

$$\begin{aligned} \mathbf{v}^{(\infty)} = & Q \partial_u + D_x Q \partial_{u_x} + D_t Q \partial_{u_t} + D_x^2 Q \partial_{u_{xx}} \\ & + D_x D_t Q \partial_{u_{xt}} + D_t^2 Q \partial_{u_{tt}} + \dots \end{aligned} \quad (21)$$

where D_x and D_t are total derivative w.r.t x and t , respectively [5].

Then, for $s = 3$, (21) is an infinitesimal local symmetry of the (1) if and only if

$$\begin{aligned} \mathbf{v}^{(\infty)} \left(u_{xt} - \alpha u - \frac{1}{3} \beta (u^3)_{xx} \right) &= 0, \\ u_{xt} &= \alpha u + \frac{1}{3} \beta (u^3)_{xx}, \\ u_{x^2 t} &= D_x \left(\alpha u + \frac{1}{3} \beta (u^3)_{xx} \right), \\ u_{xt^2} &= D_t \left(\alpha u + \frac{1}{3} \beta (u^3)_{xx} \right), \\ &\vdots \\ u_{xt^3} &= D_t^2 \left(\alpha u + \frac{1}{3} \beta (u^3)_{xx} \right), \end{aligned} \quad (22)$$

which leads to a polynomial of u_{x^5} and u_{t^5} , with functional coefficients of

$$Q(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{x^3}, u_{t^3}, u_{x^4}, u_{t^4}) \quad (23)$$

and its derivatives. All of its coefficients must be zero. This leads to a system of 5 linear determining PDEs:

$$\begin{aligned} \beta^4 u_{xx}^8 Q_{u_{t^4}} + Q_{u_{x^4}, u_{t^4}} &= 0, \\ &\vdots \\ u_x u_{tt} Q_{u, u_t} + \dots + u_x u_{t^4} Q_{u, u_{t^3}} &= 0. \end{aligned} \quad (24)$$

Therefore, the most general third-order characteristic function Q is

$$\begin{aligned} Q = & (c_1 t + c_2) u_t + 3c_1 u - c_1 x u_x + c_3 u_{t^3} - c_3 \beta^3 u_{xx}^6 u_{x^3} \\ & - \frac{3}{2} c_3 \alpha \beta^2 u_x u_{xx}^4 - (c_3 \beta \alpha^2 u_x^2 - c_5) u_x + \frac{c_4 u_{x^3}}{\sqrt{2\beta u_{x^3}^2 + \alpha}}, \end{aligned} \quad (25)$$

where c_1, \dots, c_5 are arbitrary constants. There is not any non-trivial second or forth-order characteristics. Thus, we prove that

Theorem *The most general third-order infinitesimal local symmetry generator of SPE (1) is a \mathbb{R} -linear combination of following five vector fields $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of (10) and*

$$\begin{aligned}\mathbf{v}_4 &= \frac{u_{x^3}}{\sqrt{2\beta u_{x^3}^2 + \alpha}} \partial_u, \\ \mathbf{v}_5 &= \left(u_{x^3} - \beta^3 u_{xx}^6 u_{x^3} - \frac{3}{2} \alpha \beta^2 u_x u_{xx}^4 - \alpha^2 \beta u_x^3 \right) \partial_u.\end{aligned}\tag{26}$$

There is not any non-trivial second or forth-order infinitesimal local symmetry generators.

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